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GROUPS WITH SUPER-EXPONENTIAL SUBGROUP GROWTH

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We show that, if the subgroup growth of a finitely generated (abstract or profinite) group G is super-exponential, then every finite group occurs as a quotient of a finite index subgroup of G. The proof involves techniques from finite permutation groups, and depends on the Classification of Finite Simple Groups.

1. Introduction

Let $a_n = a_n(G)$ denote the number of subgroups of index n in a finitely generated group G. The sequence $\{a_n\}$, referred to as the *subgroup growth* of G, has been the subject of extensive research in the past decade. The subgroup growth of free groups had already been determined by M. Hall in 1949 [7]. The subgroup growth of nilpotent groups was studied in [6], where rationality results for related zeta functions were established. Finitely generated groups of polynomial subgroup growth were characterized by Lubotzky, Mann and Segal [11], [14], [12]. Certain types of intermediate subgroup growth (i.e. strictly between polynomial and exponential) are studied in [9], [20], [13]. See also the survey paper [10]. Virtually nothing is known about the other end of the spectrum, namely, about groups whose subgroup growth is super-exponential. These include nonabelian free groups [17] and surface groups. Recall that surface groups are fundamental groups of closed surfaces S of genus $g \geq 1$; if $g \geq 2$ and S is orientable, or if $g \geq 3$ and S is non-orientable, then it follows from [16] that the corresponding fundamental groups have super-exponential subgroup growth.

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More generally, it can be shown that finitely presented groups on d generators with at most $d-2$ relators (in particular, one-relator groups on at least three generators) have super-exponential subgroup growth. Indeed, it is shown in [3] that these groups have finite index subgroups which project onto a free group of rank two. It would be interesting to know how close groups with super-exponential subgroup growth are to free groups. Theorem 1.1 below provides some indication, showing in particular, that such a group generates the variety of all groups. We need the following.

Definition. Let G be an infinite group. We say that a finite simple group T is an *upper composition factor* of G, if T is a composition factor of some finite quotient G/N of G.

Theorem 1.1. *Let G be a finitely generated group with super-exponential subgroup growth. Then infinitely many alternating groups A_n occur as upper composition factors of G. Consequently, every finite group is obtained as a quotient of a finite index subgroup of G.*

Remark. This result is also valid for profinite groups. Recall that a profinite group is a compact Hausdorff topological group whose open subgroups form a base for the neighbourhoods of the identity; these groups are exactly those obtained as inverse limits of finite groups. Let G be a profinite group. Then $a_n(G)$ denotes the number of *open* subgroups of index n in G, and G is said to be finitely generated if it is generated *topologically* by finitely many elements. An upper composition factor of G is a composition factor of some quotient G/N, where N is an open normal subgroup of G. It will be clear from the proof of Theorem 1.1 that, *if G is a finitely generated profinite group, and $a_n(G)$ grows super-exponentially with n, then the upper composition factors of G include infinitely many alternating groups.*

Theorem 1.1 extends several results which were obtained in the past few years. These include a theorem of Mann, showing that the subgroup growth of soluble — or pro-soluble — groups is at most exponential [15]. This result in turn is an extension of Ilani's [8] who proved the same for pro-p groups. We note that the subgroup growth of many soluble groups is indeed exponential (the simplest example is the wreath product $C_p \wr C_\infty$). This shows that the growth assumption on Theorem 1.1 cannot be weakened.

The proof of Theorem 1.1 is surprisingly short (though not elementary). Techniques from finite permutation groups play a special role in our arguments. In particular we use Babai's $c^{\log^4 n}$ bound on the number of conjugacy classes of maximal primitive subgroups of S_n [1], as well as results from [2], [4], [5]. Babai's bound depends implicitly on the Classification of Finite Simple Groups (via Cameron's classification of the primitive groups of degree n and order $> n^{c \log n}$ [5, 6.1]). A sharper bound of the form $c^{\log^2 n}$ on the number of all primitive permutation groups of degree n will be established in a subsequent paper [19]. This bound will be applied to other aspects of subgroup growth, such as counting maximal subgroups.

Finally, we note that Theorem 1.1 has a quantitative version, showing that groups of 'large' exponential subgroup growth have alternating upper composition factors of 'large' degree (see Theorem 2.4 below for the precise formulation).

We refer the reader to Wielandt's book [21] and to Cameron's survey [5] for background and basic concepts of finite permutation groups.

2. Proofs

We start with a rough bound on the number of primitive permutation groups of degree n which still suffices for our purpose.

Lemma 2.1. (CFSG) *S_n has at most $c^{\sqrt{n}\log^2 n}$ conjugacy classes of primitive subgroups.*

Proof. Recall that a primitive subgroup of S_n is called *maximal* if it is maximal (with respect to inclusion) among the primitive groups other than S_n or A_n. By Lemma 2.5 of [1], S_n has at most $c^{\log^4 n}$ conjugacy classes of maximal primitive subgroups. Let M_i $(i \in I)$ be a set of representatives for these conjugacy classes. By Lemma 7.3 of [4], every primitive subgroup of S_n can be generated by $c \log n$ elements. Hence M_i has at most $|M_i|^{c \log n}$ primitive subgroups.

Since every primitive subgroup $\neq S_n$, A_n can be extended to a maximal one, we see that the number of conjugacy classes of primitive subgroups of S_n other than S_n and A_n is bounded above by $B := \sum_{i \in I} |M_i|^{c \log n}$.

Now, by Cameron [5] we have $|M_i| \leq c^{\sqrt{n}\log n}$ (see also [1, p.151]). Hence

$$B \leq |I| c^{\sqrt{n}\log n \cdot c \log n} \leq c^{\log^4 n} c^{c\sqrt{n}\log^2 n} \leq c_1^{\sqrt{n}\log^2 n}$$

for a suitable constant c_1, The result follows. ∎

As mentioned in the introduction, the above bound can be significantly reduced [19].

Let us say that a collection $\mathcal{F}$ of finite groups is a *class with restricted composition factors*, if it is a class of finite groups all composition factors of which belong to a fixed set of finite simple groups. These are precisely those classes of finite groups which are closed under taking quotients, normal subgroups, and extensions. Examples include all finite groups, all finite soluble groups, all finite groups without alternating composition factors of degree $> k$. The classes of the last type will be used in the proof of Theorem 1.1.

Let $\mathcal{F}$ be a class with restricted composition factors. A transitive subgroup $G \subseteq S_n$ will be called a *maximal transitive $\mathcal{F}$-subgroup* if $G \in \mathcal{F}$ and whenever $G \subset H \subseteq S_n$ we have $H \notin \mathcal{F}$.

Corollary 2.2. (CFSG) (i) *S_n has at most $c^{\sqrt{n}\log^2 n}$ conjugacy classes of wreath products of primitive subgroups.*

(ii) *Consequently, if $\mathcal{F}$ is a class with restricted composition factors, then S_n has at most $c^{\sqrt{n}\log^2 n}$ conjugacy classes of maximal transitive $\mathcal{F}$-subgroups.*

Proof. The argument is similar to the proof of Lemma 3.2 (ii) of [18].

We have to count the subgroups of the form

$$G = G_1 \wr G_2 \wr \ldots \wr G_t,$$

where $t \geq 1$, $n = n_1 \cdot \ldots \cdot n_t$, and for each $i = 1, \ldots, t$ the group G_i is a primitive subgroup of S_{n_i}.

It is shown by an easy induction that the number of choices for $t, n_1, \ldots, n_t$ is at most n^2, which is negligible. We may therefore assume that these parameters are fixed. The conjugacy classes of $G_1, \ldots, G_t$ in $S_{n_1}, \ldots, S_{n_t}$ respectively determine the conjugacy class of G in S_n. Using Lemma 2.1 above we conclude that, given t, $n_1, \ldots, n_t$, the number of choices for G up to conjugacy cannot exceed

$$\prod_{i=1}^{t} c^{\sqrt{n_i}\log^2 n_i} \leq c^{\sqrt{n}\log^2 n}.$$

Part (i) follows.

Part (ii) is now proved by observing that a maximal transitive $\mathcal{F}$-subgroup of S_n is a wreath product of primitive groups. ∎

Given $\mathcal{F}$, let $\mathrm{Conj}_{\mathcal{F}}(n)$ be the number of conjugacy classes of maximal transitive $\mathcal{F}$-subgroups of S_n, and let $\mathrm{Ord}_{\mathcal{F}}(n)$ be the maximal order of such a subgroup. Let us say that an infinite group G is *an $\mathcal{F}$-group* if all finite images of G belong of $\mathcal{F}$.

Lemma 2.3. *With the above notation, let G be a d-generated $\mathcal{F}$-group. Then*

$$a_n(G) \leq n \cdot \mathrm{Conj}_{\mathcal{F}}(n) \cdot \mathrm{Ord}_{\mathcal{F}}(n)^{d-1}.$$

Proof. It is well known that $a_n(G)$ equals the number of homomorphisms $\varphi: G \to S_n$ with transitive image divided by $(n-1)!$. The image of any such homomorphism is a transitive $\mathcal{F}$-subgroup of S_n, which can be extended to a maximal transitive $\mathcal{F}$-subgroup. In order to bound $a_n(G)$ it therefore suffices to count homomorphisms from G to M, where M ranges over all maximal transitive $\mathcal{F}$-subgroups of S_n. Fix a conjugacy class C of such subgroups M and let m be the order of the subgroups in C. Then $M \in C$ can be chosen in no more than $|S_n : m| = n!/m$ ways, and given M the number of homomorphisms from G to M is bounded by m^d. It follows that

$$\sum_{M\in C} |\mathrm{Hom}\,(G, M)| \leq n! \cdot m^{d-1} \leq n! \cdot \mathrm{Ord}_{\mathcal{F}}(n)^{d-1}.$$

Summing up over all the $\mathrm{Conj}_{\mathcal{F}}(n)$ possibilities for the conjugacy class C we obtain the required bound. ∎

Let $k \geq 6$ be a fixed positive integer and let $\mathcal{F}_k$ denote the class of finite groups not involving the alternating groups A_m $(m > k)$ as composition factors.

We now apply the above result to the class $\mathcal{F}_k$.

By Lemma 2.1 of Babai–Cameron–Pálfy [2] we have

$$\mathrm{Ord}_{\mathcal{F}_k}(n) \leq k^{n-1}.$$

Furthermore, we have

$$\mathrm{Conj}_{\mathcal{F}_k}(n) \leq c^{\sqrt{n}\log^2 n},$$

by Lemma 2.2.

Applying Lemma 2.3 we conclude that, for a finitely generated $\mathcal{F}_k$-group G we have

$$a_n(G) \leq n \cdot c^{\sqrt{n}\log^2 n} \cdot (k^{n-1})^{d-1} \leq (c_1 k^{d-1})^n \tag{1}$$

where c_1 is an absolute constant and $d = d(G)$ is the minimal number of generators of G. Therefore the subgroup growth of finitely generated $\mathcal{F}_k$-groups is at most exponential. In other words, if G has super-exponential subgroup growth, then G involves infinitely many alternating groups A_k as upper composition factors. Since every finite groups can be embedded in a sufficiently large alternating groups, Theorem 1.1 follows.

We now derive a quantitative version of the main result, which also applies to groups of exponential subgroup growth. With each finitely generated group G we associate an invariant $\gamma(G) \leq \infty$ as follows:

$$\gamma(G) = \{\limsup a_n(G)^{1/n}\}^{1/(d(G)-1)},$$

if $d(G) > 1$, and $\gamma(G) = 1$ if G is cyclic. Assuming G does not have alternating upper composition factors of degree exceeding k (where $k \geq 6$), we see, using (1), that $\gamma(G) \leq k$. This gives rise to the following.

Theorem 2.4. *Let G be a finitely generated group, and let $\gamma(G)$ be as above. If $\gamma(G) = \infty$ then G has alternating upper composition factors of arbitrarily large degrees. If $6 < \gamma(G) < \infty$ then G has an alternating upper composition factor of degree $\geq \gamma(G)$.*

We conclude this paper with a few problems.

Problem 1. Let G be a finitely generated group of super-exponential subgroup growth. Does it follow that G has a finite index subgroup with a nonabelian free quotient?

In order to state the next problem we need some definitions. Let G be a group and let $f : \mathbb{N} \to \mathbb{N}$ be a function. We say that G has *growth type* f if there are constants b, $c>0$ such that

$$a_n(G) \leq f(n)^c \quad \text{for all} \quad n,$$

and

$$a_n(G) \geq f(n)^b \quad \text{for infinitely many} \quad n.$$

If G is free of rank $d>1$, then $a_n(G) \sim n \cdot (n!)^{d-1}$ [17] and so G has growth type $n!$. If follows that $n!$ is the maximal growth type of finitely generated groups. We call this growth type *factorial.*

Problem 2. Is there a finitely generated group whose growth type is strictly between exponential and factorial?

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